# AN APPLICATION OF HOCHSCHILD COHOMOLOGY TO THE MODULI OF SUBALGEBRAS OF THE FULL MATRIX RING

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Abstract. Let $\mathrm{Mold}_{n,d}$ be the moduli of $d$-dimensional subalgebras of the full matrix ring of degree $n$ over $\mathbb{Z}$. We describe the dimension of the Zariski tangent space $T_x\mathrm{Mold}_{n,d}$ and the smoothness of $\mathrm{Mold}_{n,d} \to \mathbb{Z}$ at a point $x$ of $\mathrm{Mold}_{n,d}$ by using Hochschild cohomology. We also calculate several examples of Hochschild cohomology $H^i(A, \mathrm{M}_n(k)/A)$ for $k$-subalgebras $A$ of $\mathrm{M}_n(k)$ over a field $k$.

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## 1. Introduction

Let $\mathrm{Mold}_{n,d}$ be the moduli of $d$-dimensional subalgebras of the full matrix ring of degree $n$ over $\mathbb{Z}$. More precisely, $\mathrm{Mold}_{n,d}$ is the moduli of rank $d$ molds of degree $n$ (for details, see Definition 3 and Proposition 4). In this paper, we describe the dimension of the Zariski tangent space $T_x\mathrm{Mold}_{n,d}$ and the smoothness of $\mathrm{Mold}_{n,d} \to \mathbb{Z}$ at a point $x$ of $\mathrm{Mold}_{n,d}$ by using Hochschild cohomology. We can apply these results to describing the moduli of molds. For example, we obtain

$$\mathrm{Mold}_{3,3}^{\mathrm{non-comm}} \cong ((\mathbb{P}^2_{\mathbb{Z}} \times \mathbb{P}^2_{\mathbb{Z}}) \setminus \Delta) \coprod ((\mathbb{P}^2_{\mathbb{Z}} \times \mathbb{P}^2_{\mathbb{Z}}) \setminus \Delta)$$

in Theorem 33. We have not yet found elementary proofs of Theorem 33 without using Hochschild cohomology. The long proof of Theorem 33 will be shown in [5].

We also calculate several examples of Hochschild cohomology $H^i(A, \mathrm{M}_n(k)/A)$ for $k$-subalgebras $A$ of $\mathrm{M}_n(k)$ over a field $k$. It is important to calculate $H^i(A, \mathrm{M}_n(k)/A)$ for investigating $\mathrm{Mold}_{n,d}$. It seems to us that $H^i(A, \mathrm{M}_n(k)/A)$ is easier to calculate than $H^i(A, A)$, because $H^i(A, \mathrm{M}_n(k)/A)$ often vanishes as in Theorem 22. This is one of the reason why $\mathrm{Mold}_{n,d}$ is easier to investigate than the moduli of algebras in the sense of Gabriel ([2]). There exist 26 equivalence classes of $k$-subalgebras of $\mathrm{M}_3(k)$ for any algebraically closed field $k$ ([4, Theorem 2] and [5]). We will calculate $H^i(A, \mathrm{M}_n(k)/A)$ for each $k$-subalgebras $A$ of $\mathrm{M}_3(k)$ in [6].

The organization of this paper is as follows. In Section 2, we define Hochschild cohomology and the moduli $\mathrm{Mold}_{n,d}$ of molds. In Section 3, we calculate the dimension of the Zariski tangent space of $\mathrm{Mold}_{n,d}$ at $x$ by Hochschild cohomology. In Section 4, we describe the smoothness of the morphism $\mathrm{Mold}_{n,d} \to \mathbb{Z}$. In Section 5, we introduce several examples of Hochschild cohomology $H^i(A, \mathrm{M}_n(k)/A)$ for $k$-subalgebras $A$ of $\mathrm{M}_n(k)$ over a field $k$.

The detailed version of this paper will be submitted for publication elsewhere.

## 2. Preliminaries

In this section, we define Hochschild cohomology and the moduli $\mathrm{Mold}_{n,d}$ of molds. These objects are the main characters in this paper.

**Definition 1.** Let $A$ be an associative algebra over a commutative ring $R$. Let $M$ be an $A$-bimodule. Assume that $A$ is projective over $R$. Let $A^e := A \otimes_R A^{op}$ be the enveloping algebra of $A$. For $A$-bimodules $A$ and $M$, we can regard them as left $A^e$-modules. We define the $i$-th *Hochschild cohomology group* $H^i(A, M)$ as $\mathrm{Ext}^i_{A^e}(A, M)$.

**Proposition 2.** *Let $R$, $A$, and $M$ be as above. We can calculate $H^i(A, M)$ by taking the cohomology groups of the bar complex $(C^i(A, M), d^i)_{i\in\mathbb{Z}}$ which is given by*

$$C^i(A, M) := \begin{cases} \mathrm{Hom}_R(A^{\otimes i}, M) & (i \geq 0) \\ 0 & (i < 0) \end{cases}$$

*and $d^i : C^i(A, M) \to C^{i+1}(A, M)$ $(i \geq 0)$ defined by*

$$\begin{aligned} &d^i(f)(a_1 \otimes a_2 \otimes \cdots \otimes a_{i+1}) \\ &\qquad := a_1 f(a_2 \otimes \cdots \otimes a_{i+1}) + \sum_{j=1}^{i} (-1)^j f(a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{i+1}) \\ &\qquad\qquad + (-1)^{i+1} f(a_1 \otimes a_2 \otimes \cdots \otimes a_i) a_{i+1} \end{aligned}$$

*for $f \in C^i(A, M)$. Here the tensor products are over $R$.*

For introducing the moduli of subalgebras of the full matrix ring, we define molds on schemes.

**Definition 3** ([3, Definition 1.1])**.** Let $X$ be a scheme. A subsheaf of $\mathcal{O}_X$-algebras $\mathcal{A} \subseteq \mathrm{M}_n(\mathcal{O}_X)$ is said to be a *mold* of degree $n$ on $X$ if $\mathcal{A}$ and $\mathrm{M}_n(\mathcal{O}_X)/\mathcal{A}$ are locally free sheaves on $X$. We denote by $\mathrm{rank}\mathcal{A}$ the rank of $\mathcal{A}$ as a locally free sheaf on $X$. For a commutative ring $R$, we say that an $R$-subalgebra $A \subseteq \mathrm{M}_n(R)$ is a *mold* of degree $n$ over $R$ if $A$ is a mold of degree $n$ on $\mathrm{Spec}R$. (See also [4, Remark 4].)

**Proposition 4** ([3, Definition and Proposition 1.1], [4, Proposition 5])**.** *The following contravariant functor is representable by a $\mathbb{Z}$-scheme $\mathrm{Mold}_{n,d}$.*

$$\begin{array}{cccccc} \mathrm{Mold}_{n,d} & : & (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ & & X & \mapsto & \{\ \mathcal{A} \mid \ \mathcal{A} : \textit{rank } d \textit{ mold of degree } n \textit{ on } X\ \} \end{array}$$

*Moreover, $\mathrm{Mold}_{n,d}$ is a closed subscheme of the Grassmann scheme $\mathrm{Grass}(d, n^2)$.*

Here we give examples of $\mathrm{Mold}_{n,d}$.

**Example 5.** [3, Example 1.1] In the case $n = 2$, we have

$$\begin{aligned} \mathrm{Mold}_{2,1} &= \mathrm{Spec}\mathbb{Z}, \\ \mathrm{Mold}_{2,2} &= \mathbb{P}^2_{\mathbb{Z}}, \\ \mathrm{Mold}_{2,3} &= \mathbb{P}^1_{\mathbb{Z}}, \\ \mathrm{Mold}_{2,4} &= \mathrm{Spec}\mathbb{Z}. \end{aligned}$$

**Example 6** ([4, Example 7], [5]). Let $n = 3$. If $d = 1$ or $d \geq 6$, then

$$\begin{aligned}
\mathrm{Mold}_{3,1} &= \mathrm{Spec}\mathbb{Z}, \\
\mathrm{Mold}_{3,6} &= \mathrm{Flag} := \mathrm{GL}_3/\{(a_{ij}) \in \mathrm{GL}_3 \mid a_{ij} = 0 \text{ for } i > j\}, \\
\mathrm{Mold}_{3,7} &= \mathbb{P}^2_{\mathbb{Z}} \coprod \mathbb{P}^2_{\mathbb{Z}}, \\
\mathrm{Mold}_{3,8} &= \emptyset, \\
\mathrm{Mold}_{3,9} &= \mathrm{Spec}\mathbb{Z}.
\end{aligned}$$

In [4], we introduced the following theorems.

**Theorem 7** ([4, Theorem 17], [5]).

$$\mathrm{Mold}_{3,2} \cong \mathbb{P}^2_{\mathbb{Z}} \times \mathbb{P}^2_{\mathbb{Z}}.$$

**Theorem 8** ([4, Theorem 34], [5]). *The moduli* $\mathrm{Mold}_{3,3}$ *has the following irreducible decomposition:*

$$\mathrm{Mold}_{3,3} = \overline{\mathrm{Mold}^{\mathrm{reg}}_{3,3}} \cup \overline{\mathrm{Mold}^{\mathrm{S}_2}_{3,3}} \cup \overline{\mathrm{Mold}^{\mathrm{S}_3}_{3,3}}.$$

## 3. Zariski tangent space

Let $\mathcal{A}$ be the universal mold on $\mathrm{Mold}_{n,d}$. For a point $x \in \mathrm{Mold}_{n,d}$, let $\mathcal{A}(x) \subseteq \mathrm{M}_n(k(x))$ be the corresponding mold to $x$, where $k(x)$ is the residue field of $x$. In this section, we calculate the dimension of the Zariski tangent space of $\mathrm{Mold}_{n,d}$ at $x$ by the Hochschild cohomology $H^1(\mathcal{A}(x), \mathrm{M}_n(k(x))/\mathcal{A}(x))$.

Let $A$ be a $k$-subalgebra of $\mathrm{M}_n(k)$ over a field $k$. We define $\mathrm{Der}_k(A, \mathrm{M}_n(k)/A)$ by

$$\mathrm{Der}_k(A, \mathrm{M}_n(k)/A) := \{f \in \mathrm{Hom}_k(A, \mathrm{M}_n(k)/A) \mid f(ab) = af(b) + f(a)b \text{ for } a, b \in A\}.$$

**Proposition 9** ([6]). *Let* $T_x\mathrm{Mold}_{n,d}$ *be the Zariski tangent space of* $\mathrm{Mold}_{n,d}$ *at* $x$. *There exists an isomorphism*

$$T_x\mathrm{Mold}_{n,d} \cong \mathrm{Der}_{k(x)}(\mathcal{A}(x), \mathrm{M}_n(k(x))/\mathcal{A}(x)).$$

*Proof.* The Zariski tangent space $T_x\mathrm{Mold}_{n,d}$ consists of $k(x)[\epsilon]/(\epsilon^2)$-valued points of $\mathrm{Mold}_{n,d}$ mapping the closed point to $x$. We can easily check the statement. □

For a $k$-subalgebra $A$ of $\mathrm{M}_n(k)$, let us define $d : \mathrm{M}_n(k) \to \mathrm{Der}_k(A, \mathrm{M}_n(k)/A)$ by

$$d(X)(a) := [X, a] = Xa - aX \mod A$$

for $X \in \mathrm{M}_n(k)$ and for $a \in A$. It is easy to check that $d(X) \in \mathrm{Der}_k(A, \mathrm{M}_n(k)/A)$.

**Proposition 10** ([6]). *There exists an isomorphism*

$$H^1(A, \mathrm{M}_n(k)/A) \cong \mathrm{Der}_k(A, \mathrm{M}_n(k)/A)/\mathrm{Im}\, d.$$

*Proof.* Let us consider the bar complex

$$0 \to C^0(A, \mathrm{M}_n(k)/A) \xrightarrow{d^0} C^1(A, \mathrm{M}_n(k)/A) \xrightarrow{d^1} C^2(A, \mathrm{M}_n(k)/A) \to \cdots .$$

Note that $\mathrm{Ker} d^1 = \mathrm{Der}_k(A, \mathrm{M}_n(k)/A) \supseteq \mathrm{Im}\, d^0 = \mathrm{Im}\, d$. Hence we have $H^1(A, \mathrm{M}_n(k)/A) \cong \mathrm{Der}_k(A, \mathrm{M}_n(k)/A)/\mathrm{Im}\, d$. $\square$

Let $N(A) := \{X \in \mathrm{M}_n(k) \mid [X, a] \in A \text{ for any } a \in A\}$. The $k$-linear map $d : \mathrm{M}_n(k) \to \mathrm{Der}_k(A, \mathrm{M}_n(k)/A)$ induces a $k$-linear map $\overline{d} : \mathrm{M}_n(k)/A \to \mathrm{Der}_k(A, \mathrm{M}_n(k)/A)$. Then we have the following theorem:

**Theorem 11** ([6]). *There exists the following exact sequence*

$$0 \to N(A)/A \to \mathrm{M}_n(k)/A \xrightarrow{\overline{d}} \mathrm{Der}_k(A, \mathrm{M}_n(k)/A) \to H^1(A, \mathrm{M}_n(k)/A) \to 0.$$

*In particular,*

$$\dim_{k(x)} T_x \mathrm{Mold}_{n,d} = \dim_{k(x)} H^1(\mathcal{A}(x), \mathrm{M}_n(k(x))/\mathcal{A}(x)) + n^2 - \dim_{k(x)} N(\mathcal{A}(x))$$

*for* $x \in \mathrm{Mold}_{n,d}$.

## 4. Smoothness

In this section, we consider the smoothness of the morphism $\mathrm{Mold}_{n,d} \to \mathbb{Z}$ at a point $x$ of $\mathrm{Mold}_{n,d}$. By using $H^2(\mathcal{A}(x), \mathrm{M}_n(k(x))/\mathcal{A}(x))$, we can describe the smoothness of $\mathrm{Mold}_{n,d} \to \mathbb{Z}$.

Let $(\widetilde{R}, \widetilde{m}, k)$ be an Artin local ring. Let $I$ be an ideal of $\widetilde{R}$ such that $\widetilde{m} I = 0$. Set $R := \widetilde{R}/I$ and $m := \widetilde{m}/I$. Then $(R, m, k)$ is also an Artin local ring. Denote by $\pi : \widetilde{R} \to R$ the canonical projection. Let $s : R \to \widetilde{R}$ be a set theoretical section of $\pi$.

Assume that $A \subseteq \mathrm{M}_n(R)$ is a rank $d$ mold, that is, $A$ is an $R$-subalgebra of $\mathrm{M}_n(R)$ such that $\mathrm{M}_n(R)/A$ is projective and $\mathrm{rank}_R A = d$. Let us consider the question "Is there a lift $\widetilde{A} \in \mathrm{Mold}_{n,d}(\widetilde{R})$ of $A$?" In other words, is there an $\widetilde{R}$-subalgebra $\widetilde{A} \subseteq \mathrm{M}_n(\widetilde{R})$ such that $\mathrm{M}_n(\widetilde{R})/\widetilde{A}$ is $\widetilde{R}$-projective and $\widetilde{A} \otimes_{\widetilde{R}} R = A$? If it always exists, the morphism $\mathrm{Mold}_{n,d} \to \mathbb{Z}$ is (formally) smooth. Hence we need to consider when there exists such a lift $\widetilde{A} \in \mathrm{Mold}_{n,d}(\widetilde{R})$ of $A$.

Let us take a basis $a_1, a_2, \ldots, a_{n^2}$ of $\mathrm{M}_n(R)$ over $R$ such that $a_1, a_2, \ldots, a_d$ is a basis of $A$ over $R$. For $1 \leq i \leq n^2$, choose a lift $S(a_i) \in \mathrm{M}_n(\widetilde{R})$ of $a_i$ for $1 \leq i \leq n^2$. Then we define $S : \mathrm{M}_n(R) \to \mathrm{M}_n(\widetilde{R})$ by $S(\sum_{i=1}^{n^2} r_i a_i) = \sum_{i=1}^{n^2} s(r_i) S(a_i)$ for $\sum_{i=1}^{n^2} r_i a_i \in \mathrm{M}_n(R)$. Note that $S : \mathrm{M}_n(R) \to \mathrm{M}_n(\widetilde{R})$ does not necessarily coincide with the map given by applying $s : R \to \widetilde{R}$ to each entries of matrices in $\mathrm{M}_n(R)$.

Let us define an $R$-linear map $c' : A \otimes_R A \to \mathrm{M}_n(I) \cong \mathrm{M}_n(k) \otimes_k I$ by

$$c'(\sum_{1 \leq i,j \leq d} r_{ij} a_i \otimes a_j) = \sum_{1 \leq i,j \leq d} s(r_{ij})(S(a_i a_j) - S(a_i) S(a_j))$$

for $r_{ij} \in R$. Note that $I$ is a finite-dimensional $k$-vector space, since $mI = 0$ and $I$ is a finitely generated ideal of $R$. Set $A_0 := A \otimes_R k \subseteq \mathrm{M}_n(k)$. Since $A = \oplus_{i=1}^d Ra_i$, we can write $A_0 = \oplus_{i=1}^d k\overline{a}_i$, where $\overline{a}_i := (a_i \mod m)$. We denote by $c''$ the composition

$$A \otimes_R A \xrightarrow{c'} \mathrm{M}_n(k) \otimes_k I \to (\mathrm{M}_n(k)/A_0) \otimes_k I.$$

It is easy to see that $c'' : A \otimes_R A \to (\mathrm{M}_n(k)/A_0) \otimes_k I$ goes through $A_0 \otimes_k A_0$. Then $c : A_0 \otimes_k A_0 \to (\mathrm{M}_n(k)/A_0) \otimes_k I$ is induced by $c''$. Note that $c : A_0 \otimes_k A_0 \to (\mathrm{M}_n(k)/A_0) \otimes_k I$ is a cocycle in $C^2(A_0, (\mathrm{M}_n(k)/A_0) \otimes_k I)$. Here $(\mathrm{M}_n(k)/A_0) \otimes_k I$ is an $A_0$-bimodule by $a \cdot (\overline{X} \otimes x) \cdot b = \overline{aXb} \otimes x$ for $\overline{X} \otimes x \in (\mathrm{M}_n(k)/A_0) \otimes_k I$ and for $a, b \in A_0$.

Then we can obtain the following results (for proofs, see [6]).

**Proposition 12** ([6]). *The cohomology class* $[c] \in H^2(A_0, (\mathrm{M}_n(k)/A_0) \otimes_k I)$ *is independent from the choices of* $s : R \to \widetilde{R}$, $a_1, \ldots, a_{n^2} \in \mathrm{M}_n(R)$, *and* $S(a_1), \ldots, S(a_{n^2}) \in \mathrm{M}_n(\widetilde{R})$.

**Proposition 13** ([6]). *Let* $(R, m, k)$, $(\widetilde{R}, \widetilde{m}, k)$, $I$, *and* $A_0$ *be as above. Let* $A \in \mathrm{Mold}_{n,d}(R)$. *There exists* $\widetilde{A} \in \mathrm{Mold}_{n,d}(\widetilde{R})$ *such that* $\widetilde{A} \otimes_{\widetilde{R}} R = A$ *if and only if the cohomology class* $[c]$ *is zero in* $H^2(A_0, (\mathrm{M}_n(k)/A_0) \otimes_k I)$.

**Theorem 14** ([6]). *Let* $x \in \mathrm{Mold}_{n,d}$. *If* $H^2(\mathcal{A}(x), \mathrm{M}_n(k(x))/\mathcal{A}(x)) = 0$, *then the canonical morphism* $\mathrm{Mold}_{n,d} \to \mathbb{Z}$ *is smooth at* $x$.

*Remark* 15. Even if $H^2(\mathcal{A}(x), \mathrm{M}_n(k(x))/\mathcal{A}(x)) \neq 0$, the morphism $\mathrm{Mold}_{n,d} \to \mathbb{Z}$ may be smooth at $x \in \mathrm{Mold}_{n,d}$. For details, see Remark 30.

## 5. Hochschild cohomology $H^*(A, \mathrm{M}_n(k)/A)$

Let $A$ be a $k$-subalgebra of $\mathrm{M}_n(k)$ over a field $k$. We calculate several examples of Hochschild cohomology groups $H^i(A, \mathrm{M}_n(k)/A)$.

Let $Q$ be a finite quiver. Denote by $Q_0$ and $Q_1$ the sets of vertices and arrows of $Q$, respectively. For each oriented path $\alpha$ of $Q$, we denote by $h(\alpha)$ and $t(\alpha)$ the head and the tail of $\alpha$, respectively. Let $RQ$ be the path algebra over a commutative ring $R$. We define the *arrow ideal* $F$ as the two-sided ideal of $RQ$ generated by the paths of positive length of $Q$.

**Definition 16.** A two-sided ideal of $I$ of $RQ$ is called *admissible* if $F^n \subset I \subset F$ for a positive integer $n$ and $F/I$ is an $R$-free module which has an $R$-basis consisting of oriented paths.

For an admissible ideal $I$, set $\Lambda = RQ/I$ and $r = F/I$. Denote by $E$ the $R$-subalgebra of $\Lambda$ generated by $Q_0$. We can use the following result in [1] for calculating Hochschild cohomology.

**Proposition 17** ([1, Proposition 1.2]). *Let* $M$ *be a* $\Lambda$*-bimodule. The Hochschild cohomology* $R$*-modules* $H^i(\Lambda, M)$ *are the cohomology groups of the complex of* $E$*-bimodules*

$$0 \to M^E \xrightarrow{\delta^0} \mathrm{Hom}_{E^e}(r, M) \xrightarrow{\delta^1} \mathrm{Hom}_{E^e}(r \otimes_E r, M) \xrightarrow{\delta^2} \cdots$$
$$\cdots \xrightarrow{\delta^{i-1}} \mathrm{Hom}_{E^e}(r^{\otimes i}, M) \xrightarrow{\delta^i} \mathrm{Hom}_{E^e}(r^{\otimes i+1}, M) \xrightarrow{\delta^{i+1}} \cdots,$$

*where the tensor products are over $E$ and*

$$\begin{array}{rcl} M^E & = & \{m \in M \mid sm = ms \text{ for each } s \in Q_0\} \\ \delta^0(m)(x) & := & xm - mx \text{ for } m \in M^E \text{ and for } x \in r, \\ \delta^i(f)(x_1 \otimes \cdots \otimes x_{i+1}) & := & x_1 f(x_2 \otimes \cdots \otimes x_{i+1}) \\ & & + \sum_{j=1}^{i} (-1)^j f(x_1 \otimes \cdots \otimes x_j x_{j+1} \otimes \cdots \otimes x_{i+1}) \\ & & + (-1)^{i+1} f(x_1 \otimes \cdots \otimes x_i) x_{i+1}. \end{array}$$

*Remark* 18. Set $r^{\otimes 0} := E$. Then $\mathrm{Hom}_{E^e}(r^{\otimes 0}, M) = M^E$. Hence the complex above can be written by $\{\mathrm{Hom}_{E^e}(r^{\otimes n}, E), \delta^n\}$.

**Definition 19.** Let $Q$ be a finite quiver without oriented cycles. We say that $Q$ is *ordered* if there exists no oriented path other than $\alpha$ joining $t(\alpha)$ to $h(\alpha)$ for each arrow $\alpha \in Q_1$.

**Definition 20.** Let $Q$ be an ordered quiver. Let $I$ be the two-sided ideal of $RQ$ generated by

$$\left\{ \gamma - \delta \in RQ \;\middle|\; \begin{array}{c} \gamma \text{ and } \delta \text{ are oriented paths with} \\ h(\gamma) = h(\delta) \text{ and } t(\gamma) = t(\delta) \end{array} \right\}.$$

We call $\Lambda = RQ/I$ an *incidence $R$-algebra*. Note that $I$ is an admissible ideal.

For an ordered quiver $Q$, set $n := |Q_0|$. For $a, b \in Q_0$, we define $a \geq b$ if $a = b$ or there exists an oriented path $\alpha$ such that $t(\alpha) = a$ and $h(\alpha) = b$. Then $(Q_0, \geq)$ is a partially ordered set (i.e. poset). Let $\Lambda := RQ/I$ be the incidence algebra associated to $Q$. For $a \geq b$, let $e_{b,a}$ be the equivalence class of oriented paths from $a$ to $b$ in $\Lambda$. We can write $\Lambda = \oplus_{a \geq b} Re_{b,a}$. Fix a numbering on $Q_0$. By regarding $e_{b,a}$ as $E_{ba}$, $\Lambda$ can be considered as an $R$-subalgebra of $\mathrm{M}_n(R) = \oplus_{a,b \in Q_0} RE_{ba}$, where $E_{ba}$ is the matrix unit. We can write $E = \oplus_{a \in Q_0} Re_{a,a}$ and $E^e = E \otimes_R E^{op} = \oplus_{a,b \in Q_0} Re_{a,a} \otimes e_{b,b}$. We also have $r = F/I = \oplus_{a > b} Re_{b,a}$. (In the sequel, we denote $E_{ba} \in \mathrm{M}_n(R)$ by $e_{b,a}$ for simplicity.)

**Lemma 21** ([6])**.** *For $i \geq 0$, $\mathrm{Hom}_{E^e}(r^{\otimes i}, \mathrm{M}_n(R)/\Lambda) = 0$.*

*Proof.* As $E$-bimodules, $r^{\otimes i}$ is isomorphic to $\oplus_{s_0 > s_1 > \cdots > s_i} Re_{s_i, s_0}$. On the other hand, $\mathrm{M}_n(R)/\Lambda \cong \oplus_{a \not\geq b} Re_{b,a}$. Hence we have

$$\mathrm{Hom}_{E^e}(r^{\otimes i}, \mathrm{M}_n(R)/\Lambda) \cong \oplus_{s_0 > s_1 > \cdots > s_i,\, a \not\geq b} \mathrm{Hom}_{E^e}(Re_{s_i,s_0}, Re_{b,a}).$$

Since $\mathrm{Hom}_{E^e}(Re_{s_i,s_0}, Re_{b,a}) \cong e_{s_i,s_i}(Re_{b,a})e_{s_0,s_0} = 0$, $\mathrm{Hom}_{E^e}(r^{\otimes i}, \mathrm{M}_n(R)/\Lambda) = 0$. □

In this case, the complex $\{\mathrm{Hom}_{E^e}(r^{\otimes n}, E), \delta^n\}$ is zero. Summarizing the discussion above, we have the following theorem.

**Theorem 22** ([6])**.** *Let $Q$ be an ordered quiver with $n = |Q_0|$. Let $\Lambda$ be the incidence algebra associated to $Q$. Then $H^i(\Lambda, \mathrm{M}_n(R)/\Lambda) = 0$ for $i \geq 0$.*

**Example 23** ([6])**.** Let us consider the following quiver $Q$

$$1 \longleftarrow 2 \longleftarrow 3 \longleftarrow \cdots \longleftarrow n.$$

Let $\Lambda = RQ/I$ be the incidence algebra associated to $Q$ over a commutative ring $R$. Then $\Lambda = \oplus_{1\le i\le j\le n} Re_{i,j}$. We can regard $\Lambda$ as the upper triangular matrix ring

$$\mathcal{B}_n(R) := \{(a_{ij}) \in \mathrm{M}_n(R) \mid a_{ij} = 0 \text{ for } i > j\}.$$

By Theorem 22,

$$H^i(\mathcal{B}_n(R), \mathrm{M}_n(R)/\mathcal{B}_n(R)) = 0$$

for $i \ge 0$.

We introduce several examples without proofs.

**Definition 24.** Let $n_1, n_2, \ldots, n_s$ be positive integers with $\sum_{i=1}^{s} n_i = n$. We define the $R$-subalgebra $\mathcal{P}_{n_1,n_2,\ldots,n_s}(R)$ of $\mathrm{M}_n(R)$ over a commutative ring $R$ by

$$\mathcal{P}_{n_1,n_2,\ldots,n_s}(R) = \{(a_{ij}) \in \mathrm{M}_n(R) \mid a_{ij} = 0 \text{ if } \sum_{k=1}^{t} n_k < i \le \sum_{k=1}^{t+1} n_k \text{ and } j \le \sum_{k=1}^{t} n_k\}.$$

In particular, $\mathcal{P}_{1,1,\ldots,1}(R) = \mathcal{B}_n(R)$.

**Proposition 25** ([6])**.** *Let $R$ be a commutative ring. Let $\mathcal{P}_{n_1,n_2,\ldots,n_s}(R)$ be as in Definition 24. Then $H^i(\mathcal{P}_{n_1,n_2,\ldots,n_s}(R), \mathrm{M}_n(R)/\mathcal{P}_{n_1,n_2,\ldots,n_s}(R)) = 0$ for $i \ge 0$.*

Let $R$ be a commutative ring, and let $\mathrm{D}_n(R) := \{(a_{ij}) \in \mathrm{M}_n(R) \mid a_{ij} = 0 \text{ for } i \ne j\} \subset \mathrm{M}_n(R)$. In other words, $\mathrm{D}_n(R)$ is the $R$-subalgebra of diagonal matrices in $\mathrm{M}_n(R)$. Let $Q$ be the quiver consisting of $n$ vertices and no arrows. Then $D_n(R)$ is isomorphic to the incidence algebra associated to $Q$. By Theorem 22, we obtain:

**Proposition 26** ([6])**.** *For $i \ge 0$, $H^i(\mathrm{D}_n(R), \mathrm{M}_n(R)/\mathrm{D}_n(R)) = 0$.*

**Definition 27.** Let $R$ be a commutative ring. We define $x \in \mathrm{M}_n(R)$ by

$$x = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Let $\mathrm{J}_n(R)$ be the $R$-subalgebra of $\mathrm{M}_n(R)$ generated by $x$. Then $\mathrm{J}_n(R) \cong R[x]/(x^n)$ as $R$-algebras.

**Proposition 28** ([6])**.** *Let $\mathrm{J}_n(R)$ be as above. Then*

$$H^i(\mathrm{J}_n(R), \mathrm{M}_n(R)/\mathrm{J}_n(R)) = \begin{cases} R^{n-1} \oplus \mathrm{Ann}(n) & (i: \textit{ even }) \\ R^{n-1} \oplus R/nR & (i: \textit{ odd }), \end{cases}$$

*where* $\mathrm{Ann}(n) := \{a \in R \mid an = 0\}$.

**Corollary 29** ([6])**.** *For a field $k$,*

$$H^i(\mathrm{J}_n(k), \mathrm{M}_n(k)/\mathrm{J}_n(k)) = \begin{cases} k^{n-1} & (\mathrm{ch}(k) \nmid n) \\ k^n & (\mathrm{ch}(k) \mid n). \end{cases}$$

*Remark* 30. Although $H^2(\mathrm{J}_n(k), \mathrm{M}_n(k)/\mathrm{J}_n(k)) \neq 0$, $\mathrm{Mold}_{n,n}$ is smooth at $\mathrm{J}_n(k)$. Indeed, $J_n(k)$ is contained in $\mathrm{Mold}_{n,n}^{\mathrm{reg}} := \overline{\mathrm{Mold}_{n,n}^{\mathrm{D_n}}} \subseteq \mathrm{Mold}_{n,n}$ and $\mathrm{Mold}_{n,n}^{\mathrm{reg}}$ is smooth over $\mathbb{Z}$.

We introduce an example of applications of Hochschild cohomology to describing the moduli of molds. For a commutative ring $R$, set

$$S_2(R) := \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & c \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b, c \in R \right\} \text{ and } S_3(R) := \left\{ \begin{pmatrix} a & 0 & c \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b, c \in R \right\}.$$

**Proposition 31** ([6]). *For $A = \mathrm{S}_2(R)$ or $\mathrm{S}_3(R)$,*

$$H^i(A, \mathrm{M}_3(R)/A) = \begin{cases} R^2 & (i = 0) \\ 0 & (i > 0). \end{cases}$$

**Definition 32.** We define the non-commutative subalgebras part $\mathrm{Mold}_{n,d}^{\mathrm{non-comm}}$ of $\mathrm{Mold}_{n,d}$ by

$$\mathrm{Mold}_{n,d}^{\mathrm{non-comm}} := \{x \in \mathrm{Mold}_{n,d} \mid \mathcal{A}(x) \text{ is not commutative } \},$$

where $\mathcal{A}(x) \subseteq \mathrm{M}_n(k(x))$ is the corresponding mold to $x$. Note that $\mathrm{Mold}_{n,d}^{\mathrm{non-comm}}$ is an open subscheme of $\mathrm{Mold}_{n,d}$.

**Theorem 33** ([5]). *Let $V$ be a free sheaf of rank 3 on $\mathrm{Spec}\mathbb{Z}$. Denote by $\mathbb{P}_*(V)$ and $\mathbb{P}^*(V)$ the projective spaces consisting of rank 1 and rank 2 subbundles of $V$ on $\mathrm{Spec}\mathbb{Z}$, respectively. Let $\mathrm{Mold}_{3,3}^{\mathrm{S_2}}$ and $\mathrm{Mold}_{3,3}^{\mathrm{S_3}}$ be the subschemes of $\mathrm{Mold}_{3,3}$ consisting of subalgebras of type $S_2$ and $S_3$, respectively. Then*

$$\mathrm{Mold}_{3,3}^{\mathrm{S_2}} \cong (\mathbb{P}^*(V) \times \mathbb{P}^*(V)) \setminus \Delta(\mathbb{P}^*(V))$$

*and*

$$\mathrm{Mold}_{3,3}^{\mathrm{S_3}} \cong (\mathbb{P}_*(V) \times \mathbb{P}_*(V)) \setminus \Delta(\mathbb{P}_*(V)),$$

*where $\Delta(\mathbb{P}^*(V))$ and $\Delta(\mathbb{P}_*(V))$ are the diagonals of $\mathbb{P}^*(V) \times \mathbb{P}^*(V)$ and $\mathbb{P}_*(V) \times \mathbb{P}_*(V)$, respectively. Moreover, we have*

$$\mathrm{Mold}_{3,3}^{\mathrm{non-comm}} = \mathrm{Mold}_{3,3}^{\mathrm{S_2}} \coprod \mathrm{Mold}_{3,3}^{\mathrm{S_3}} \cong ((\mathbb{P}^2_{\mathbb{Z}} \times \mathbb{P}^2_{\mathbb{Z}}) \setminus \Delta) \coprod ((\mathbb{P}^2_{\mathbb{Z}} \times \mathbb{P}^2_{\mathbb{Z}}) \setminus \Delta),$$

*where $\Delta$ is the diagonal of $\mathbb{P}^2_{\mathbb{Z}} \times \mathbb{P}^2_{\mathbb{Z}}$.*

By Proposition 31, $\mathrm{Mold}_{3,2}^{\mathrm{S_2}}$ and $\mathrm{Mold}_{3,2}^{\mathrm{S_3}}$ are smooth over $\mathbb{Z}$. We need the result that the local ring is reduced at each point of $\mathrm{Mold}_{3,2}^{\mathrm{S_2}}$ and $\mathrm{Mold}_{3,2}^{\mathrm{S_3}}$ for proving Theorem 33. The proof of Theorem 33 is a little bit long. For details, see [5].

**Corollary 34** ([5]). *Let $(R, m, k)$ be a local ring. Let $A \subseteq \mathrm{M}_3(R)$ be an $R$-subalgebra of $\mathrm{M}_3(R)$ such that $A$ and $\mathrm{M}_3(R)/A$ are $R$-projective, $\mathrm{rank}_R A = 3$, and $A \otimes_R k$ is not commutative. Then there exists $P \in \mathrm{GL}_3(R)$ such that $P^{-1}AP = \mathrm{S}_2(R)$ or $\mathrm{S}_3(R)$.*

*In other words, there exist distinct subline bunldes $L_1, L_2$ of $R^3$ or distinct rank 2 subbundles $W_1, W_2$ of $R^3$ such that*

$$A = \langle \mathrm{Hom}_R(R^3/L_1, L_2) \rangle \subset \mathrm{End}_R(R^3) = \mathrm{M}_3(R)$$

*or*

$$A = \langle \mathrm{Hom}_R(R^3/W_1, W_2) \rangle \subset \mathrm{End}_R(R^3) = \mathrm{M}_3(R).$$

*Here we regard* $\mathrm{Hom}_R(R^3/L_1, L_2)$ *and* $\mathrm{Hom}_R(R^3/W_1, W_2)$ *as the* $R$*-submodules of* $\mathrm{End}_R(R^3)$ *by* $f \mapsto \iota \circ f \circ \pi$*, where* $\iota : L_2 \to R^3$ *(or* $\iota : W_2 \to R^3$*) is the inclusion and* $\pi : R^3 \to R^3/L_1$ *(or* $\pi : R^3 \to R^3/W_1$*) is the projection. We denote by* $\langle S \rangle$ *the subalgebra of* $\mathrm{End}_R(R^3)$ *generated by a subset* $S$ *of* $\mathrm{End}_R(R^3)$*.*

## References


[1] C. Cibils, *Cohomology of incidence algebras and simplicial complexes*, Journal of Pure and Applied Algebra **56** (1989) 221–232.

[2] P. Gabriel, *Finite representation type is open*, Proceedings of the International Conference on Representations of Algebras (Carleton Univ., Ottawa, Ont., 1974), Paper No. 10, 23 pp. Carleton Math. Lecture Notes, No. 9, Carleton Univ., Ottawa, Ont., 1974.

[3] K. Nakamoto, *The moduli of representations with Borel mold*, Internat. J. Math. **25** (2014), no. 7, 1450067, 31 pp.

[4] K. Nakamoto and T. Torii, *The moduli of subalgebras of the full matrix ring of degree 3*, Proceedings of the 50th Symposium on Ring Theory and Representation Theory, 137–149, Symp. Ring Theory Represent. Theory Organ. Comm., Yamanashi, 2018.

[5] ———, *On the classification of subalgebras of the full matrix ring of degree* 3, in preparation.

[6] ———, *An application of Hochschild cohomology to the moduli of subalgebras of the full matrix ring*, in preparation.



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